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# Towards Drinfeld–Sokolov reduction for quantum groups

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## Abstract

In this paper we study the Poisson–Lie version of the Drinfeld–Sokolov reduction defined in [E. Frenkel, N. Reshetikhin, M.A. Semenov-Tian-Shansky, Drinfeld–Sokolov reduction for difference operators and deformations of  $W$ -algebras. I. The case of Virasoro algebra, *Comm. Math. Phys.* 192 (1998) 605; M.A. Semenov-Tian-Shansky, A.V. Sevostyanov, Drinfeld–Sokolov reduction for difference operators and deformations of  $W$ -algebras. II. General semisimple case, *Comm. Math. Phys.* 192 (1998) 631]. Using the bialgebra structure related to the new Drinfeld realization of affine quantum groups we describe reduction in terms of constraints. This realization of reduction admits direct quantization.

As a byproduct we obtain an explicit expression for the symplectic form associated to the twisted Heisenberg double and calculate the moment map for the twisted dressing action. For some class of infinite-dimensional Poisson–Lie groups we also prove an analogue of the Ginzburg–Weinstein isomorphism. ©2000 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

It is well known that the quantum Drinfeld–Sokolov reduction (see [9]) plays an important role in the theory of affine Lie algebras. Let  $\mathfrak{g}$  be a complex semisimple Lie algebra,  $\hat{\mathfrak{g}}$  the affinization of  $\mathfrak{g}$ . In [11,12], Feigin and Frenkel proved that the Drinfeld–Sokolov reduction associated to  $\mathfrak{g}$  describes the structure of the center of the universal enveloping algebra  $U(\hat{\mathfrak{g}})$  at the critical level of the central charge.

The Poisson–Lie version of the Drinfeld–Sokolov reduction was proposed in [13,23]. The quantization of this reduction is expected to play the same role for affine quantum groups. In this paper we obtain a realization of the Drinfeld–Sokolov reduction for Poisson–Lie groups admitting direct quantization.

Recall that in physical terms every reduction procedure consists of two steps: (1) imposing constraints, (2) choosing a cross-section of the constraint surface. To quantize a system with constraints one should quantize the underlying Poisson manifold, find quantum counterparts of the classical constraints and apply the quantum reduction procedure to the quantum system. All the known versions of this procedure require an explicit description of constraints. For the Lie–Poisson version of the Drinfeld–Sokolov reduction both the quantization of the underlying Poisson manifold and the description of constraints are nontrivial problems.

The main observation of [13,23] was that in order to perform the Poisson–Lie version of the Drinfeld–Sokolov reduction associated to a complex semisimple Lie algebra  $\mathfrak{g}$  one should introduce a new quasitriangular bialgebra structure on the loop algebra  $\mathfrak{g}((z))$  arising from natural geometric considerations. The corresponding  $r$ -matrix is obtained by adding an extra term to the standard  $r$ -matrix related to the “new Drinfeld realization” of affine quantum groups. This term is essentially elliptic and may be expressed by means of theta functions (see Section 3.2). We denote by  $\mathcal{G}$  the corresponding Poisson–Lie group. The Poisson manifold which undergoes reduction is essentially the dual Poisson–Lie group, with its standard Poisson structure being twisted by an automorphism satisfying certain conditions (see (12)). In [13], this Poisson structure is called gauge covariant. We denote the corresponding Poisson manifold by  $\mathcal{G}_p$ . This manifold is equipped with the twisted dressing action of the Poisson–Lie group  $\mathcal{G}$ .

In the theory of Poisson group actions constraints naturally appear as matrix coefficients of moment maps in the sense of Lu and Weinstein. However, the twisted dressing action of the unipotent subgroup  $\mathcal{N} = LN \subset \mathcal{G}$  used in [13,23] for reduction is not a Poisson group action. In Section 2, we define a map  $\mu_{\mathcal{N}}$  from the underlying Poisson manifold  $\mathcal{G}_p$  to the opposite unipotent group  $\overline{\mathcal{N}}$  which forms a dual pair together with the canonical projection onto the quotient  $\mathcal{G}_p/\mathcal{N}$  and serves as a substitute of the moment map.

As a matter of fact, it is not really necessary to quantize the new elliptic bialgebra structure. It is well known that at least in the finite-dimensional case quantizations of different bialgebra structures are isomorphic as algebras (see [7]). The classical counterpart of this statement for Lie–Poisson groups is called the Ginzburg–Weinstein isomorphism [14]. We show that the same statement holds for some class of infinite-dimensional Poisson–Lie groups. Applying a simple form of the Ginzburg–Weinstein isomorphism found in [1], we prove that the gauge covariant Poisson structures corresponding to the new bialgebra structure and to the one related to Drinfeld’s “new realization” of affine quantum groups are isomorphic. This allows us to use the latter for reduction. Surprisingly, the description of constraints in this realization does not contain elliptic functions. Quantum counterparts of these constraints have been defined in [25].

Remarkably, the quantum constraints are of the first class, i.e. they form a subalgebra that possesses a character fixing the values of the constraints. An appropriate reduction technique for such systems of constraints has been developed by the author in [26].

## 2. The Poisson geometry of the twisted dressing action

In this section we develop the moment map technique for the twisted dressing action. The moment map is important both to describe the constraints and to prove the existence of the Ginzburg–Weinstein isomorphism (see [1]).

Let  $G \times M \rightarrow M$  be a Poisson group action of a Poisson–Lie group  $G$  on a Poisson manifold  $M$  possessing a moment map  $\mu$ . Then  $\mu$  maps the manifold  $M$  into the dual Poisson–Lie group  $G^*$ . The most important particular example of a Poisson group action is the dressing action of a Poisson–Lie group  $G$  on the dual group  $G^*$ ; in that case the moment map is the identity mapping. For quasitriangular Poisson–Lie groups the easiest way to obtain this action is to consider  $G^*$  as the reduced Poisson manifold for the Heisenberg double  $D$  of  $G$  with respect to the left Poisson group action of  $G$ . Recall that the Heisenberg double is isomorphic to  $G \times G$  as a manifold and  $G$  acts on the product  $G \times G$  by left and right diagonal translations. The right Poisson group action of  $G$  on the Heisenberg double generates the dressing action on  $G^*$ .

For the needs of the Drinfeld–Sokolov reduction we have to twist the standard Poisson structure of  $G^*$  by an automorphism satisfying certain conditions (see (12)). This twisted Poisson structure, called the gauge covariant Poisson structure in [13], may be obtained by a reduction from the twisted Heisenberg double of  $G$ . The twisted Heisenberg double is equipped with left and right Poisson group actions of the Poisson–Lie group  $G$ . The Poisson reduction with respect to the left action of  $G$  yields a gauge covariant Poisson structure on  $G^*$ . Then the right Poisson group action of the same group generates the twisted dressing action of  $G$  on  $G^*$  called the gauge action in [13].

The Poisson structure on the twisted Heisenberg double is nondegenerate on an open dense subset. Following [2], we obtain an explicit expression for the corresponding symplectic form. Using this formula we calculate the moment maps for the right and left Poisson group actions of  $G$  on the twisted Heisenberg double. In contrast with the untwisted case, these maps are neither Poisson nor equivariant. As a consequence, we get the moment map for the gauge action of  $G$  on  $G^*$ . This result will be applied to the Drinfeld–Sokolov reduction in the next section.

### 2.1. Factorizable Lie bialgebras and their doubles

Let us recall some basic notions on Poisson–Lie groups (see [7,17,22]). Let  $G$  be a Lie group equipped with a Poisson bracket,  $\mathfrak{g}$  its Lie algebra.  $G$  is called a Poisson–Lie group if the multiplication  $G \times G \rightarrow G$  is a Poisson map. A Poisson bracket satisfying this axiom is degenerate, and in particular, is identically zero at the unit element of the group. Linearizing this bracket at the unit element we get the structure of a Lie algebra in the space  $T_e^*G \simeq \mathfrak{g}^*$ . The pair  $(\mathfrak{g}, \mathfrak{g}^*)$  is called the tangent bialgebra of  $G$ .  $(\mathfrak{g}, \mathfrak{g}^*)$  is called a *factorizable Lie bialgebra* if the following conditions are satisfied [7,17]:

1.  $\mathfrak{g}$  is equipped with a fixed nondegenerate invariant scalar product  $\langle \cdot, \cdot \rangle$ . We shall always identify  $\mathfrak{g}^*$  and  $\mathfrak{g}$  by means of this scalar product.
2. The dual Lie bracket on  $\mathfrak{g}^* \simeq \mathfrak{g}$  is given by

$$[X, Y]_* = \frac{1}{2}([rX, Y] + [X, rY]), \quad X, Y \in \mathfrak{g}, \tag{1}$$

where  $r \in \text{End } \mathfrak{g}$  is a skew symmetric linear operator (classical r-matrix).

3.  $r$  satisfies the modified classical Yang–Baxter identity:

$$[rX, rY] - r([rX, Y] + [X, rY]) = -[X, Y], \quad X, Y \in \mathfrak{g}. \tag{2}$$

Define operators  $r_{\pm} \in \text{End } \mathfrak{g}$  by

$$r_{\pm} = \frac{1}{2}(r \pm id).$$

Then the classical Yang–Baxter equation implies that  $r_{\pm}$ , regarded as a mapping from  $\mathfrak{g}^*$  into  $\mathfrak{g}$ , is a Lie algebra homomorphism. Moreover,  $r_+^* = -r_-$ , and  $r_+ - r_- = id$ .

The double of a factorizable Lie bialgebra admits the following explicit description (cf. [22], Section 2). Put  $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$  (direct sum of two copies). The mappings

$$\begin{aligned} \mathfrak{g}^* \rightarrow \mathfrak{d} & : X \mapsto (X_+, X_-), \quad X_{\pm} = r_{\pm}X, \\ \mathfrak{g} \rightarrow \mathfrak{d} & : X \mapsto (X, X), \end{aligned} \tag{3}$$

are Lie algebra embeddings. Thus we may identify  $\mathfrak{g}^*$  and  $\mathfrak{g}$  with Lie subalgebras in  $\mathfrak{d}$ . Equip  $\mathfrak{d}$  with the scalar product

$$\langle\langle (X, X'), (Y, Y') \rangle\rangle = \langle X, Y \rangle - \langle X', Y' \rangle. \tag{4}$$

**Proposition 1** (Semenov-Tian-Shansky [22], Proposition 2.1).

1.  $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$  is a Manin triple, i.e.  $\mathfrak{g}$  and  $\mathfrak{g}^*$  are isotropic subalgebras with respect to the scalar product (4).
2.  $\mathfrak{d}$  is isomorphic to the double of  $(\mathfrak{g}, \mathfrak{g}^*)$ .
3.  $(\mathfrak{d}, \mathfrak{d}^*)$  is a factorizable Lie bialgebra; the corresponding r-matrix  $r_{\mathfrak{d}} \in \text{End}(\mathfrak{g} \oplus \mathfrak{g})$  is given by

$$r_{\mathfrak{d}} = \begin{pmatrix} r & -2r_+ \\ 2r_- & -r \end{pmatrix}. \tag{5}$$

The problem of classification of solutions of the classical Yang–Baxter equation was solved by Belavin and Drinfeld in [4] (cf. also [20]). Their results may be summarized as follows.

Denote by  $\mathfrak{b}_{\pm}$  and  $\mathfrak{n}_{\mp}$  the image and the kernel of the operator  $r_{\pm}$  :  $\mathfrak{b}_{\pm} = \text{Im } r_{\pm}$ ,  $\mathfrak{n}_{\mp} = \text{Ker } r_{\pm}$ .

**Theorem 2** (Belavin–Drinfeld). *Let  $(\mathfrak{g}, \mathfrak{g}^*)$  be a factorizable Lie bialgebra. Then*

1.  $\mathfrak{b}_{\pm} \subset \mathfrak{g}$  is a Lie subalgebra; the subspace  $\mathfrak{n}_{\pm}$  is a Lie ideal in  $\mathfrak{b}_{\pm}$ ,  $\mathfrak{b}_{\pm}^{\perp} = \mathfrak{n}_{\pm}$ .
2. The map  $\theta_r : \mathfrak{b}_-/\mathfrak{n}_- \rightarrow \mathfrak{b}_+/\mathfrak{n}_+$  which sends the residue class of  $r_-(X)$ ,  $X \in \mathfrak{g}^*$ , modulo  $\mathfrak{n}_-$  to that of  $r_+(X)$  modulo  $\mathfrak{n}_+$  is a well-defined isomorphism of Lie algebras. Moreover,  $\theta_r$  is a unitary operator with respect to the induced scalar product:  $\theta_r \theta_r^* = 1$ .

Using part (2) of Theorem 2 we can describe the image of the embedding  $\mathfrak{g}^* \rightarrow \mathfrak{d}$  as follows:

$$\mathfrak{g}^* = \{(X_+, X_-) \in \mathfrak{b}_+ \oplus \mathfrak{b}_- \subset \mathfrak{d} \mid \overline{X}_+ = \theta_r(\overline{X}_-)\}, \quad \text{where } \overline{X}_{\pm} = X_{\pm} \text{ mod } \mathfrak{n}_{\pm}. \tag{6}$$

We shall also need other properties of the subalgebras  $\mathfrak{b}_\pm$ .

**Proposition 3** (Belavin and Drinfeld [4]).

1.  $\mathfrak{n}_\pm$  is an ideal in  $\mathfrak{g}^*$ .
2.  $\mathfrak{b}_\pm$  is a Lie subalgebra in  $\mathfrak{g}^*$ . Moreover  $\mathfrak{b}_\pm = \mathfrak{g}^*/\mathfrak{n}_\pm$ .
3.  $(\mathfrak{b}_\pm, \mathfrak{b}_\pm^*)$  is a sub-bialgebra of  $(\mathfrak{g}, \mathfrak{g}^*)$  and  $\mathfrak{b}_\pm^* \simeq \mathfrak{b}_\mp$ . The canonical pairing between  $\mathfrak{b}_\mp$  and  $\mathfrak{b}_\pm$  is given by

$$\langle X_\mp, Y_\pm \rangle_\pm = \langle X_\mp, r_\pm^{-1} Y_\pm \rangle, \quad X_\mp \in \mathfrak{b}_\mp; \quad Y_\pm \in \mathfrak{b}_\pm. \tag{7}$$

4.  $\mathfrak{n}_\pm^* \simeq \mathfrak{n}_\mp$  as a linear space.

**Remark 1.** If  $X_\pm \in \mathfrak{b}_\pm$ ,  $(Y_+, Y_-) \in \mathfrak{g}^*$ ,  $Y_\pm = r_\pm Y$ , then

$$\langle \langle (X_-, X_-), (Y_+, Y_-) \rangle \rangle = \langle X_-, Y \rangle = (X_-, Y_+)_+, \tag{8}$$

$$\langle \langle (X_+, X_+), (Y_+, Y_-) \rangle \rangle = \langle X_+, Y \rangle = (X_+, Y_-)_-. \tag{9}$$

### 2.2. Twisted Heisenberg double and its symplectic structure

We start the study of the twisted Heisenberg double with explicit formulae for some Poisson brackets associated with the bialgebra structure on  $\mathfrak{d}$ .

For every Lie group  $A$  with Lie algebra  $\mathfrak{a}$  we define left and right gradients  $\nabla\varphi, \nabla'\varphi \in \mathfrak{a}^*$  of a function  $\varphi \in C^\infty(A)$  by

$$\xi(\nabla\varphi(x)) = \left(\frac{d}{ds}\right)_{s=0} \varphi(e^{s\xi}x), \quad \xi(\nabla'\varphi(x)) = \left(\frac{d}{ds}\right)_{s=0} \varphi(xe^{s\xi}), \quad \xi \in \mathfrak{a}. \tag{10}$$

The Poisson bracket on  $D = G \times G$  associated with the bialgebra structure on  $\mathfrak{d}$  has the form (see [22], Section 2):

$$\{\varphi, \psi\} = -\frac{1}{2} \langle \langle r_{\mathfrak{d}} \nabla\varphi, \nabla\psi \rangle \rangle + \frac{1}{2} \langle \langle r_{\mathfrak{d}} \nabla'\varphi, \nabla'\psi \rangle \rangle. \tag{11}$$

It is well known that the bracket (11) satisfies the Jacobi identity and equips  $D$  with the structure of a Poisson–Lie group. The embeddings  $\mathfrak{g} \rightarrow \mathfrak{d}, \mathfrak{g}^* \rightarrow \mathfrak{d}$  may be extended to homomorphisms  $G \rightarrow D, G^* \rightarrow D$ . We shall identify  $G$  and  $G^*$  with the corresponding subgroups in  $D$ . By the definition of the double  $G, G^* \subset D$  are Lie–Poisson subgroups.

Let  $\sigma \in \text{Aut } G$  be an automorphism of  $G$ . We shall denote the corresponding automorphism of  $\mathfrak{g}$  by the same letter. Assume that  $\sigma$  satisfies the following conditions:

1.  $\sigma \circ r = r \circ \sigma$ .
  2.  $\langle \sigma X, \sigma Y \rangle = \langle X, Y \rangle$  for all  $X, Y \in \mathfrak{g}$ .
- (12)

We shall associate with  $\sigma$  the so-called *twisted* Poisson structure on  $D$  (see [22], Section 5). Let  $T \in \text{Aut } \mathfrak{d}$  be the automorphism of the Lie algebra  $\mathfrak{d}$  defined by

$$T = \left( \sigma^{-1} \times id \right).$$

Denote by  ${}^\sigma r_{\mathfrak{d}}$  the  $r$ -matrix  $r_{\mathfrak{d}}$  twisted by the automorphism  $T$ :

$${}^\sigma r_{\mathfrak{d}} = T r_{\mathfrak{d}} T^{-1} = \begin{pmatrix} r & -2\sigma^{-1}r_+ \\ 2r_{-\sigma} & -r \end{pmatrix},$$

and define the twisted Poisson bracket on  $D$  by

$$\{\varphi, \psi\}_\sigma = \frac{1}{2} \langle \langle r_{\mathfrak{d}} \nabla \varphi, \nabla \psi \rangle \rangle + \frac{1}{2} \langle \langle {}^\sigma r_{\mathfrak{d}} \nabla' \varphi, \nabla' \psi \rangle \rangle. \tag{13}$$

The Jacobi identity for (13) follows from the classical Yang–Baxter identity (2) for  $r_{\mathfrak{d}}$ .

The pair  $D_\sigma = (D, \{, \}_\sigma)$  is called the *twisted Heisenberg double* (for  $\sigma = id$  we get the ordinary Heisenberg double).

When restricted to an open dense subset the Poisson structure of the twisted Heisenberg double is nondegenerate. To calculate the corresponding symplectic form we need twisted factorizations on  $D$  [22].

**Proposition 4.**

1. Any element  $x \in \mathfrak{d}$  admits two unique decompositions

$$x = \eta + T\xi, \quad x = T\eta' + \xi', \quad \eta, \eta' \in \mathfrak{g}, \xi, \xi' \in \mathfrak{g}^*. \tag{14}$$

2. In an open dense subset  $D'_\sigma \subset D_\sigma$  we have the following factorizations:

$$\begin{aligned} d &= gg^{*T} = h^*h^T, \quad \text{where } d \in D'_\sigma, \\ g, h &\in G, \quad g^*, h^* \in G^*, \\ g^* &= (g_+, g_-), \quad h^* = (h_+, h_-). \end{aligned} \tag{15}$$

**Theorem 5.**  $D'_\sigma$  is a symplectic submanifold in  $D_\sigma$ . The corresponding symplectic form can be represented as follows:

$$\Omega = \langle \langle \theta_{h^*} \otimes \theta_g \rangle \rangle - \langle \langle \mu_h \otimes \mu_{g^*} \rangle \rangle, \tag{16}$$

where  $\theta_{h^*}, \theta_g(\mu_{g^*}, \mu_h)$  are the universal right-invariant (left-invariant) Maurer–Cartan forms on  $G^*$  and  $G$ , respectively; the pairing is applied to their values and lower indices indicate group variables.

**Proof.** Quite similar to that for  $\sigma = id$  (see [2], Theorem 3). □

We shall call  $D'_\sigma$  the principal symplectic leaf of  $D_\sigma$ .

One can define an action of the Drinfeld double  $D$  on the Poisson manifold  $D_\sigma$  which generalizes the well-known dressing action [19].

**Proposition 6.**

1. The actions of  $D$  on  $D_\sigma$  by right and left translations are Poisson group actions;

$$D \times D_\sigma \xrightarrow{L} D_\sigma, \quad d' \circ d = d(d'^T)^{-1}, \quad D_\sigma \times D \xrightarrow{R} D_\sigma, \quad d \circ d' = d'^{-1}d. \tag{17}$$

where  $d \in D_\sigma, d' \in D$ .

2. The actions (17) generate Poisson group actions of the Poisson–Lie subgroups  $G, G^* \subset D$ :

$$G \times D_\sigma \xrightarrow{L} D_\sigma, \tag{18}$$

$$D_\sigma \times G \xrightarrow{R} D_\sigma, \tag{19}$$

$$G^* \times D_\sigma \xrightarrow{L} D_\sigma, \quad D_\sigma \times G^* \xrightarrow{R} D_\sigma. \tag{20}$$

3. The restrictions of the actions (18)–(20) to the principal symplectic leaf  $D'_\sigma$  possess moment maps in the sense of Lu and Weinstein [16]:

$$\begin{aligned} \mu_G^L(d) &= g^*, & \mu_{G^*}^L(d) &= h, \\ \mu_G^R(d) &= h^*, & \mu_{G^*}^R(d) &= g, \end{aligned} \tag{21}$$

where  $g, g^*, h, h^*$  are given by (15).

**Remark 2.** In general, the maps (21) are neither Poisson nor equivariant.

**Proof.** Proofs of (1) and (2) are given in [19], Proposition 2.5.1. (See also [22], Proposition 5.4).

To prove (3), let us consider, for instance, the action

$$D_\sigma \times G \xrightarrow{R} D_\sigma. \tag{22}$$

Let  $X \in \mathfrak{g}$ . The corresponding vector field is

$$\hat{X}\varphi(d) = \left( \frac{d}{ds} \right)_{s=0} \varphi(e^{-sX}d). \tag{23}$$

Therefore,  $\hat{X} = (R_d)_*(-X)$ , where  $R_d$  is the right translation by  $d$ . According to the definition of the moment map [16], we have to prove that

$$\Omega(\hat{X}, \cdot) = -\langle \mu_G^{R*}(\theta), X \rangle. \tag{24}$$

Formula (15) for the twisted factorization problem implies that  $\mu_{g^*}(\hat{X}) = 0$ . Therefore, substituting  $\hat{X}$  into (16) we obtain:

$$\Omega(\hat{X}, \cdot) = -\Omega(\cdot, \hat{X}) = -\langle \theta_{h^*} \otimes \theta_g(\hat{X}) \rangle. \tag{25}$$

We also have  $\theta_g(\hat{X}) = X$ , because  $\hat{X}$  is a right-invariant vector field and  $G$  is a subgroup in  $D$ . Finally,

$$\Omega(\hat{X}, \cdot) = -\langle \theta_{h^*} \otimes X \rangle. \tag{26}$$

This completes the proof. □

Let  $B_\pm$  be the Lie subgroups in  $G$  corresponding to the Lie subalgebras  $\mathfrak{b}_\pm$ . According to part (3) of Proposition 3,  $B_\pm$  are Poisson–Lie subgroups in  $G$ .

**Proposition 7.** *The G-actions (18) and (19) induce Poisson group actions of the Poisson subgroups  $B_{\pm}$ . When restricted to the principal symplectic leaf these actions possess moment mappings in the sense of Lu and Weinstein:*

$$\mu_{B_{\mp}}^R(d) = h_{\pm}, \quad \mu_{B_{\mp}}^L(d) = g_{\pm}, \tag{27}$$

where  $h_{\pm}, g_{\pm}$  are given by (15).

**Proof.** Follows from the previous proposition and Remark 1. □

### 2.3. Gauge covariant Poisson structures

The gauge covariant Poisson structure used in [13,23] for the Drinfeld–Sokolov reduction may be obtained from the twisted Heisenberg double by the following construction ([21], Section 3; [22], Section 5).

Consider the Poisson reduction of the Poisson manifold  $D_{\sigma}$  with respect to the left action (18) of the group  $G$ . The quotient space  $G \backslash D_{\sigma}$  may be identified with  $G$ , the projection map  $p : D_{\sigma} \rightarrow G$  is given by

$$p : (x, y) \mapsto x^{\sigma} (y)^{-1}.$$

Under this identification the reduced Poisson bracket on  $G$  is given by

$$\{\varphi, \psi\}_{\sigma} = \langle r \nabla \varphi, \nabla \psi \rangle + \langle r \nabla' \varphi, \nabla' \psi \rangle - 2 \langle r_{+}^{\sigma} \nabla' \varphi, \nabla \psi \rangle - 2 \langle r_{-}^{\sigma} \nabla \varphi, \nabla' \psi \rangle, \tag{28}$$

where  $r_{+}^{\sigma} = \sigma \circ r_{+}$ ,  $r_{-}^{\sigma} = r_{-} \circ \sigma^{-1}$ .

Denote by  $G_{\sigma} = (G, \{, \}_{\sigma})$  the manifold  $G$  equipped with Poisson bracket (28).

Then the right action (19) gives rise to a Poisson group action of  $G$  on the reduced space:

$$G_{\sigma} \times G \rightarrow G_{\sigma} : g \circ L = (g^{\sigma})^{-1} L g. \tag{29}$$

From Propositions 4,6 and 7 and Theorem 5 we deduce the following properties of the reduced Poisson manifold and the gauge action (29).

**Proposition 8.**

1. *Elements  $L \in G$  admitting a twisted factorization*

$$L = L_{+}^{\sigma} L_{-}^{-1}, \quad (L_{+}, L_{-}) \in G^* \tag{30}$$

*form an open dense subset  $G'_{\sigma}$  in  $G$ . This factorization is unique in a neighborhood of the unit element.*

2.  *$G'_{\sigma}$  is a Poisson submanifold in  $G_{\sigma}$ .*
3. *The restriction of the action (29) to  $G'_{\sigma}$  has a moment mapping given by the identity map:  $L \mapsto (L_{+}, L_{-})$ .*
4. *When restricted to  $G'_{\sigma}$  the induced actions of the Poisson–Lie subgroups  $B_{\mp} \subset G$  have moment maps given by*

$$\mu_{B_{\mp}}(L) = L_{\pm}. \tag{31}$$



### 3. Moment map and Drinfeld–Sokolov reduction

In this section we obtain different descriptions of the Drinfeld–Sokolov reduction for Poisson–Lie groups (see [13,23] for the definition of the reduction). Using the moment map technique developed in the previous section, we adapt the reduction procedure for quantization. First, we find a system of constraints for the reduction. This allows us to describe the reduced space by means of Dirac’s technique (see [6]). Then we show that different bialgebra structures can be used for the reduction. The most important particular case corresponds to the bialgebra structure related to the new Drinfeld realization of affine quantum groups. It is this description of reduction that is important for quantization.

#### 3.1. Drinfeld–Sokolov reduction for Poisson–Lie groups

Recall the construction of the Drinfeld–Sokolov reduction for Poisson–Lie groups.

Let  $G$  be a connected simply connected finite-dimensional complex semisimple Lie group,  $\mathfrak{g}$  its Lie algebra. Fix a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  and let  $\Delta$  be the set of roots of  $(\mathfrak{g}, \mathfrak{h})$ . Choose an ordering in the root system; let  $\Delta_+$  be the system of positive roots,  $\{\alpha_1, \dots, \alpha_l\}$ ,  $l = \text{rank } \mathfrak{g}$ , the set of simple roots and  $H_1, \dots, H_l$  the set of simple root generators of  $\mathfrak{h}$ .

Denote by  $a_{ij}$  the corresponding Cartan matrix. Let  $d_1, \dots, d_l$  be coprime positive integers such that the matrix  $b_{ij} = d_i a_{ij}$  is symmetric. There exists a unique nondegenerate invariant symmetric bilinear form  $(\cdot, \cdot)$  on  $\mathfrak{g}$  such that  $(H_i, H_j) = d_j^{-1} a_{ij}$ . It induces an isomorphism of vector spaces  $\mathfrak{h} \simeq \mathfrak{h}^*$  under which  $\alpha_i \in \mathfrak{h}^*$  corresponds to  $d_i H_i \in \mathfrak{h}$ . The induced bilinear form on  $\mathfrak{h}^*$  is given by  $(\alpha_i, \alpha_j) = b_{ij}$ .

Let  $\mathfrak{b}$  be the positive Borel subalgebra and  $\bar{\mathfrak{b}}$  the opposite Borel subalgebra; let  $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$  and  $\bar{\mathfrak{n}} = [\bar{\mathfrak{b}}, \bar{\mathfrak{b}}]$  be their nil-radicals. Let  $H = \exp \mathfrak{h}$ ,  $N = \exp \mathfrak{n}$ ,  $\bar{N} = \exp \bar{\mathfrak{n}}$ ,  $B = HN$ ,  $\bar{B} = H\bar{N}$  be the Cartan subgroup, the maximal unipotent subgroups and the Borel subgroups of  $G$  which correspond to the Lie subalgebras  $\mathfrak{h}$ ,  $\mathfrak{n}$ ,  $\bar{\mathfrak{n}}$ ,  $\mathfrak{b}$  and  $\bar{\mathfrak{b}}$ , respectively.

Let  $\mathfrak{g} = L\mathfrak{g}$  be the loop algebra; we equip it with the standard invariant bilinear form,

$$\langle X, Y \rangle = \text{Res}_{z=0} (X(z), Y(z)) dz/z. \quad (32)$$

Define  $b = L\mathfrak{b}$ ,  $\bar{b} = L\bar{\mathfrak{b}}$ ,  $n = L\mathfrak{n}$ ,  $\bar{n} = L\bar{\mathfrak{n}}$ ,  $h = L\mathfrak{h}$ , etc. The quotient algebras  $b/n$ ,  $\bar{b}/\bar{n}$  may be canonically identified with  $h$ .

Denote by  $\mathcal{G}$ ,  $\mathcal{B}$ ,  $\bar{\mathcal{B}}$ ,  $\mathcal{N}$ ,  $\bar{\mathcal{N}}$ ,  $\mathcal{H}$  the corresponding loop groups. The groups  $G$ ,  $B$ ,  $\bar{B}$ ,  $N$ ,  $\bar{N}$ ,  $H$  will be identified with the subgroups of constant loops.

Introduce an operator  ${}^\theta r \in \text{End } \mathfrak{g}$  by

$${}^\theta r = P_{\bar{n}} - P_n + r^0, \quad r^0 = \frac{1+\theta}{1-\theta} P_h, \quad (33)$$

where  $P_h$  is the projector onto  $h$  in the direct sum

$$\mathfrak{g} = \mathfrak{n} + \bar{\mathfrak{n}} + \mathfrak{h},$$

and  $\theta \in \text{End } \mathfrak{h}$  is a unitary automorphism with respect to the scalar product (32) such that  $\det(\theta - 1) \neq 0$ . The operator (33) satisfies the classical Yang–Baxter equation. Therefore, the Lie algebra  $\mathfrak{g}$  is equipped with the structure of a factorizable Lie bialgebra. Moreover, in the notation of Theorem 2, we have  $\mathfrak{b}_- = \mathfrak{b}$ ,  $\mathfrak{b}_+ = \bar{\mathfrak{b}}$ ,  $\mathfrak{n}_- = \mathfrak{n}$ ,  $\mathfrak{n}_+ = \bar{\mathfrak{n}}$  and  $\theta_r = \theta$ .

Fix  $p \in \mathbb{C}$ ,  $|p| < 1$ , and let  $D_p$  be the automorphism of  $\mathcal{G}$  defined by  $(D_p g)(z) = g(pz)$ . We shall denote the corresponding automorphism of the loop algebra  $\mathfrak{g}$  by the same letter.

Note that  $D_p$  preserves the scalar product (32). Assume also that  $\theta$  commutes with  $D_p$ . Then the automorphism  $D_p$  and the r-matrix (33) satisfy conditions (12). Therefore, we can endow the group  $\mathcal{G}$  with Poisson bracket (28) for  $r = {}^\theta r$ ,  $\sigma = D_p$ . We denote the corresponding Poisson manifold by  $\mathcal{G}_{p,\theta}$ . This manifold is equipped with the Poisson group action (29) of the Poisson–Lie group  $\mathcal{G}$ :

$$g \circ L = (g^{D_p})^{-1} Lg. \tag{34}$$

Let  $W$  be the Weyl group of  $(\mathfrak{g}, \mathfrak{h})$ ; we shall denote a representative of  $w \in W$  in  $G$  by the same letter. We also denote  $w(g) = wgw^{-1}$  for any  $g \in G$ . Let  $s_1, \dots, s_l$  be the reflections which correspond to the simple roots; let  $s = s_1 s_2 \dots s_l$  be a Coxeter element.

The Drinfeld–Sokolov reduction for Poisson–Lie groups is a Poisson reduction for the Poisson manifold  $\mathcal{G}_{p,\theta}$ , where  $\theta = D_p \cdot s$ , with respect to the gauge action (34) of the unipotent group  $\mathcal{N} \subset \mathcal{G}$ . The result of the reduction is a Poisson submanifold  $\mathcal{S} \subset \mathcal{G}_{p,D_p \cdot s} / \mathcal{N}$  which has a nice geometric description.

Denote  $\mathcal{N}' = \{v \in \mathcal{N}; svs^{-1} \in \bar{\mathcal{N}}\}$ ,  $\mathcal{N}'' = \{v \in \mathcal{N}; svs^{-1} \in \bar{\mathcal{N}}\}$ ,  $\mathcal{M}^s = \mathcal{N}s^{-1}\mathcal{N}$ ,  $\mathcal{M}^s = \mathcal{N}s^{-1}\mathcal{N}$ .

**Theorem 9** (Semenov-Tian-Shansky and Sevostyanov [23]).

1. The gauge action (34) of the group  $\mathcal{N}$  leaves the cell  $\mathcal{M}^s \subset \mathcal{G}_{p,\theta}$  invariant. The action of  $\mathcal{N}$  on  $\mathcal{M}^s$  is free and  $\mathcal{S} = \mathcal{N}'s^{-1}$  is a cross-section of this action.
2. The subgroup  $\mathcal{N}$  is admissible in  $\mathcal{G}$  and hence  $\mathcal{N}$ -invariant functions form a Poisson subalgebra in the Poisson algebra  $C^\infty(\mathcal{G}_{p,\theta})$ .
3. The quotient space  $\mathcal{M}^s / \mathcal{N} = \mathcal{S}$  is a Poisson submanifold in  $\mathcal{G}_{p,\theta} / \mathcal{N}$  if and only if the endomorphism  $\theta$  is given by  $\theta = s \cdot D_p$ , where  $s \in W$  is a Coxeter element.

To simplify the notation we shall denote the Poisson manifold  $\mathcal{G}_{p,D_p \cdot s}$  by  $\mathcal{G}_p$ . The r-matrix  ${}^{s \cdot D_p} r$  that enters the definition of the Poisson structure of this manifold has the form:

$${}^{s \cdot D_p} r = P_{\bar{\mathfrak{n}}} - P_{\mathfrak{n}} + r^0, \quad r^0 = \frac{1 + s \cdot D_p}{1 - s \cdot D_p} P_{\mathfrak{h}}. \tag{35}$$

From (6) it follows that the factorization problem (30) for the manifold  $\mathcal{G}'_p$  amounts to the relations:

$$L = L_+^{D_p} L_-^{-1}, \quad L \in \mathcal{G}'_p, (L_+, L_-) \in \mathcal{G}^*, L_\pm = h_\pm n_\pm, h_\pm \in \mathcal{H},$$

$$h_- = s(h_+^{D_p}), n_+ \in \bar{\mathcal{N}}, n_- \in \mathcal{N}. \tag{36}$$

We shall also consider the case of constant loops. The restriction of operator (33) to constant loops defines an  $r$ -matrix  ${}^\theta r \in \text{End } \mathfrak{g}$ , where  $\theta \in \text{End } \mathfrak{h}$ . We denote the corresponding Poisson–Lie group by  $G$ . Let  $G_\theta$  be the manifold  $G$  equipped with Poisson bracket (28), where  $r = {}^\theta r$ ,  $\sigma = id$ . Then the action of  $G$  on  $G_\theta$  by conjugations is a Poisson group action. We can formulate the finite-dimensional version of the previous theorem.

**Corollary 10.**

1. *The action of the group  $N$  on  $G_\theta$  by conjugations leaves the cell  $M^s \subset G_\theta$  invariant. The action of  $N$  on  $M^s$  is free and  $S = N's^{-1}$  is a cross-section of this action.*
2. *The subgroup  $N$  is admissible in  $G$  and hence  $N$ -invariant functions form a Poisson subalgebra in the Poisson algebra  $C^\infty(G_\theta)$ .*
3. *The quotient space  $M^s/N=S$  is a Poisson submanifold in  $G_\theta/N$  if and only if the endomorphism  $\theta$  is given by  $\theta = s$ , where  $s \in W$  is the Coxeter element*

From the results of [23] (see Section 4, Theorem 4.10) it follows that the Poisson structure of the reduced Poisson manifold  $S$  may be described using a Poisson surjection  $m : H \rightarrow S$  called the generalized Miura transform. For  $\sigma = id$  formula 4.3 in [23] implies that the Poisson structure of the manifold  $H$  is trivial. Therefore the Poisson bracket on  $S$  equals to zero identically.

Note also that  $N' \subset N$  is an abelian subgroup,  $\dim N^s = l$  (see [5]). So we arrive to

**Proposition 11.** *The algebra of regular functions on the reduced space  $M^s/N = N's^{-1}$  is a polynomial algebra with  $l$  generators; the reduced Poisson structure is trivial.*

3.2. *The structure of the  $r$ -matrix*

In this section we show that  $r$ -matrix (35) is elliptic. Namely, we shall express the kernel of its “Cartan” component  $r^0$  by means of theta functions.

The kernel of operator  $r^0$  is given by a formal power series

$$r^0\left(\frac{z}{w}\right) = \sum_{n=-\infty}^{\infty} \frac{1 + p^n s}{1 - p^n s} \left(\frac{z}{w}\right)^n, \tag{37}$$

where  $s$  regarded as an element of  $\mathfrak{h} \otimes \mathfrak{h} \simeq \text{End } \mathfrak{h}$ .

Let  $h$  be the Coxeter number of  $\mathfrak{g}$ . Since  $s^h = id$ ,  $r^0(z)$  may be represented in the following form:

$$r^0(z) = \sum_{m=0}^{h-1} s^m \sum_{n=-\infty}^{\infty} \frac{p^{nm}}{1 - p^{nh}} z^n. \tag{38}$$

According to Theorem 2.5 [23], this formal power series satisfies the following functional equation:

$$r^0(z) = sr^0(pz) + s\delta(pz) + \delta(z), \quad \delta(z) = \sum_{n=-\infty}^{\infty} z^n. \tag{39}$$

By successive application of this identity to  $r^0(z)$  we obtain

$$r^0(z) = r^0(p^h z) + \delta(z) + \delta(p^h z) + 2 \sum_{k=1}^{h-1} s^k \delta(p^k z). \tag{40}$$

Therefore, if the formal power series (38) has a nontrivial domain of convergence it will define an *End*  $h$ -valued elliptic function. The delta functions in (40) indicate singularities of this function. We shall show that this is indeed the case by comparing the series (38) with the Fourier expansion of an elliptic function. Recall that every such function may be expressed via theta functions.

The standard theta function  $\theta_t$  is defined by the relation

$$\theta_t(u) = c \prod_{n=-\infty}^{\infty} (1 - t^{2n-1} e^{2\pi i u})(1 - t^{2n-1} e^{-2\pi i u}),$$

$$c = \prod_{n=1}^{\infty} (1 - t^{2n}), \quad t = e^{\pi i \xi}, \quad \text{Im } \xi > 0 \tag{41}$$

and satisfies the functional equations:

$$\theta_t(u + 1) = \theta_t(u), \quad \theta_t(u + \xi) = -t^{-1} e^{-2\pi i u} \theta_t(u). \tag{42}$$

The only zeroes of  $\theta_t$  are located at the points  $m + (n + 1/2)\xi$ ,  $m, n \in \mathbb{Z}$ .

In the stripe  $|\text{Im } u| \leq (1/2) \text{Im } \xi$ ,  $u \notin \{m \pm (1/2)\xi, m \in \mathbb{Z}\}$  the ratio  $\theta'_t/\theta_t(u)$  has the following Fourier expansion (see [3]):

$$\frac{\theta'_t}{\theta_t}(u) = \frac{2\pi}{i} \sum_{n=-\infty}^{\infty} \frac{t^n}{1 - t^{2n}} e^{2\pi i n u} \tag{43}$$

Put  $z = e^{2\pi i u}$ .

**Proposition 12.** For  $u \in \mathbb{R} \setminus \mathbb{Z}$  the formal power series (38) coincides with the Fourier expansion of the following elliptic function:

$$r^0(u) = \frac{i}{2\pi} \left( 2 \sum_{m=1}^{h-1} s^m \frac{\theta'_t}{\theta_t} \left( u + \xi \left( \frac{m}{h} - \frac{1}{2} \right) \right) + \frac{\theta'_t}{\theta_t} \left( u - \frac{\xi}{2} \right) + \frac{\theta'_t}{\theta_t} \left( u + \frac{\xi}{2} \right) \right),$$

where  $p$  and  $t = e^{\pi i \xi}$  are related by  $p^{h/2} = t$ .

**Proof.** Follows from formula (43) and expression (38) for  $r^0$ . □

### 3.3. Dual pairs for some admissible actions and the Drinfeld–Sokolov reduction

Now we shall describe the Drinfeld–Sokolov reduction in terms of constraints. For group actions admitting a moment map the constraints imposed during reduction are given by matrix coefficients of the moment map. The gauge action (34) of the subgroup  $\mathcal{N}$  is not a Poisson group action and hence it does not have a moment map in the usual sense.

Nevertheless, it is possible to define an analogue of the moment map in this more general situation.

Let  $A \times M \rightarrow M$  be a right Poisson group action of a Poisson–Lie group  $A$  on a Poisson manifold  $M$ . A subgroup  $K \subset A$  is called *admissible* if the set  $C^\infty(M)^K$  of  $K$ -invariants is a Poisson subalgebra in  $C^\infty(M)$ .

**Proposition 13** (Semenov-Tian-Shansky [21], Theorem 6). *Let  $(\mathfrak{a}, \mathfrak{a}^*)$  be the tangent Lie bialgebra of  $A$ . A connected Lie subgroup  $K \subset A$  with Lie algebra  $\mathfrak{k} \subset \mathfrak{a}$  is admissible if  $\mathfrak{k}^\perp \subset \mathfrak{a}^*$  is a Lie subalgebra.*

Let  $A \times M \rightarrow M$  be a right Poisson group action of a Poisson–Lie group  $A$  on a manifold  $M$ . Suppose that this action possesses a moment mapping  $\mu : M \rightarrow A^*$ . Let  $K$  be an admissible subgroup in  $A$ . Denote by  $\mathfrak{k}$  the Lie algebra of  $K$ . Assume that  $\mathfrak{k}^\perp \subset \mathfrak{a}^*$  is a Lie subalgebra in  $\mathfrak{a}^*$ . Suppose also that there is a splitting  $\mathfrak{a}^* = \mathfrak{t} \oplus \mathfrak{k}^\perp$ , and that  $\mathfrak{t}$  is a Lie subalgebra in  $\mathfrak{a}^*$ . Then the linear space  $\mathfrak{k}^*$  is naturally identified with  $\mathfrak{t}$ . Assume that  $A^*$  is the semidirect product of the Lie subgroups  $K^\perp, T$  corresponding to the Lie algebras  $\mathfrak{k}^\perp, \mathfrak{t}$ , respectively. Suppose that  $K^\perp$  is a connected subgroup in  $A^*$ . Fix the decomposition  $A^* = K^\perp T$  and denote by  $\pi_{K^\perp}, \pi_T$  the projections onto  $K^\perp$  and  $T$  in this decomposition.

**Theorem 14.** *Define a map  $\bar{\mu} : M \rightarrow T$  by*

$$\bar{\mu} = \pi_T \mu.$$

Then

1.  $\bar{\mu}^*(C^\infty(T))$  is a Poisson subalgebra in  $C^\infty(M)$ , and hence one can equip  $T$  with a Poisson structure such that  $\bar{\mu} : M \rightarrow T$  is a Poisson map.
2. Moreover, the algebra  $C^\infty(M)^K$  is the centralizer of  $\bar{\mu}^*(C^\infty(T))$  in the Poisson algebra  $C^\infty(M)$ . In particular, if  $M/K$  is a smooth manifold the maps

$$\begin{array}{ccc}
 & M & \\
 \swarrow \pi & & \searrow \bar{\mu} \\
 M/K & & T
 \end{array}, \tag{44}$$

form a dual pair.

**Proof.**

1. First, by Theorem 4.9 in [16] there exists a Poisson bracket on  $A^*$  such that  $\mu : M \rightarrow A^*$  is a Poisson map. Moreover, we can choose this bracket to be the sum of the standard Poisson–Lie bracket of  $A^*$  and of a left invariant bivector on  $A^*$ . Denote by  $A_M^*$  the manifold  $A^*$  equipped with this Poisson structure. Now observe that  $T$  is identified with the quotient  $K^\perp A_M^*$ , where  $K^\perp$  acts on  $A_M^*$  by multiplications from the left. Therefore, to prove part (1) of the proposition it suffices to show that  $K^\perp$ -invariant functions on  $A_M^*$  form a Poisson subalgebra in  $C^\infty(A_M^*)$ .

Observe that since  $A^*$  is a Poisson–Lie group and the Poisson structure of  $A_M^*$  is obtained from that of  $A^*$  by adding a left-invariant term, the action of  $A^*$  on  $A_M^*$  by multiplications from the left is a Poisson group action. Note also that  $K^\perp$  is a connected

subgroup in  $A^*$  and  $(\mathfrak{k}^\perp)^\perp \cong \mathfrak{k}$  is a Lie subalgebra in  $\mathfrak{a}$ . Therefore by Proposition 13,  $K^\perp$  is an admissible subgroup in  $A^*$ . Therefore,  $K^\perp$ -invariant functions on  $A_M^*$  form a Poisson subalgebra in  $C^\infty(A_M^*)$ , and hence  $\overline{\mu}^*(C^\infty(T))$  is a Poisson subalgebra in  $C^\infty(M)$ . This proves part (1).

2. By the definition of the moment map, we have

$$L_{\hat{X}}\varphi = \langle \mu^*(\theta_{A^*}), X \rangle (\xi_\varphi), \tag{45}$$

where  $X \in \mathfrak{a}$ ,  $\hat{X}$  is the corresponding vector field on  $M$  and  $\xi_\varphi$  is the Hamiltonian vector field of  $\varphi \in C^\infty(M)$ . Since  $A^*$  is the semidirect product of  $K^\perp$  and  $T$  the pullback of the right-invariant Maurer–Cartan form  $\mu^*(\theta_{A^*})$  may be represented as follows:

$$\mu^*(\theta_{A^*}) = Ad(\pi_{K^\perp}\mu)(\overline{\mu}^*\theta_T) + (\pi_{K^\perp}\mu)^*\theta_{K^\perp},$$

where  $Ad(\pi_{K^\perp}\mu)(\overline{\mu}^*\theta_T) \in \mathfrak{t}$ ,  $(\pi_{K^\perp}\mu)^*\theta_{K^\perp} \in \mathfrak{k}^\perp$ .

Now, let  $X \in \mathfrak{k}$ . Then  $\langle (\pi_{K^\perp}\mu)^*\theta_{K^\perp}, X \rangle = 0$  and formula (45) takes the form:

$$L_{\hat{X}}\varphi = \langle Ad(\pi_{K^\perp}\mu)(\overline{\mu}^*\theta_T), X \rangle (\xi_\varphi) = \langle Ad(\pi_{K^\perp}\mu)(\theta_T), X \rangle (\overline{\mu}_*(\xi_\varphi)). \tag{46}$$

Since  $Ad(\pi_{K^\perp}\mu)$  is a nondegenerate transformation,  $L_{\hat{X}}\varphi = 0$  for every  $X \in \mathfrak{k}$  if and only if  $\overline{\mu}_*(\xi_\varphi) = 0$ , i.e. a function  $\varphi \in C^\infty(M)$  is  $K$ -invariant if and only if  $\{\varphi, \overline{\mu}^*(\psi)\} = 0$  for every  $\psi \in C^\infty(T)$ . This completes the proof.  $\square$

Recall that the notion of dual pairs serves to describe Poisson submanifolds in the quotient  $M/K$  (see [28]). If  $M$  is symplectic, connected components of the sets  $\pi(\overline{\mu}^{-1}(x))$ ,  $x \in T$  are symplectic leaves in  $M/K$ . This allows us to give an alternative description of the Drinfeld–Sokolov reduction.

Using action (34) of the group  $\mathcal{B}$  on  $\mathcal{G}'_p$  in the setting of the previous theorem we shall construct a dual pair for the gauge action of the subgroup  $\mathcal{N} \subset \mathcal{B}$ . Observe that according to part (3) of Proposition 3  $(b, \bar{b})$  is a subalgebra of  $(g, g^*)$ . Therefore,  $\mathcal{B}$  is a Poisson subgroup in  $\mathcal{G}$ . The restriction of the  $\mathcal{B}$ -gauge action (34) to  $\mathcal{G}'_p$  has a moment map given by Proposition 8 and formula (36) for the factorization problem:

$$\mu_{\mathcal{B}}(L) = L_+, \quad L = L_+^{D_p} L_-^{-1}, \quad (L_+, L_-) \in \mathcal{G}^*. \tag{47}$$

The orthogonal complement of the  $n \subset b$  in the dual space  $\bar{b}$  coincides with the  $h \subset \bar{b}$ . Hence, by Proposition 13  $\mathcal{N}$  is an admissible subgroup in the Poisson group  $\mathcal{B}$ . Moreover, the dual group  $\bar{\mathcal{B}}$  is the semidirect product of the Lie groups  $\mathcal{H}$  and  $\bar{\mathcal{N}}$  corresponding to the Lie algebras  $n^\perp = h$  and  $\bar{n} \cong n^*$ , respectively. We conclude that all the conditions of Theorem 14 are satisfied with  $A = \mathcal{B}$ ,  $K = \mathcal{N}$ ,  $A^* = \bar{\mathcal{B}}$ ,  $T = \bar{\mathcal{N}}$ ,  $K^\perp = \mathcal{H}$ . It follows that Poisson manifold  $\mathcal{G}'_p$  possesses a dual pair formed by the canonical projection onto the quotient  $\mathcal{G}'_p/\mathcal{N}$  and the map  $\mu_{\mathcal{N}}$ ,

$$\begin{aligned} \mu_{\mathcal{N}}(L) &= n_+, \quad L = L_+^{D_p} L_-^{-1}, \quad (L_+, L_-) \in \mathcal{G}^*, \\ L_+ &= h_+ n_+, \quad h_+ \in \mathcal{H}, \quad n_+ \in \bar{\mathcal{N}}. \end{aligned} \tag{48}$$

We shall describe the reduction with the help of the map  $\mu_{\mathcal{N}}$ .

Let  $w_0 \in W$  be the longest element; let  $\tau \in \text{Aut } \Delta_+$  be the automorphism defined by  $\tau(\alpha) = -w_0 \cdot \alpha, \alpha \in \Delta_+$ . Let  $N_i \subset N$  be the 1-parameter subgroup generated by the root vector corresponding to the root  $\tau(\alpha_i)$ . Choose an element  $u_i \in N_i, u_i \neq 1$ . Then we have (see [24])  $w_0 u_i w_0^{-1} \in B s_i B$ . We may fix  $u_i$  in such a way that  $w_0 u_i w_0^{-1} \in N s_i N$ . Set  $x = u_i u_{i-1} \dots u_1$ ; then  $f := w_0 x w_0^{-1} \in N s^{-1} N \cap \bar{N}$ .

**Proposition 15.** *The set  $\mu_{\mathcal{N}}^{-1}(f)$  is an open subset in  $\mathcal{M}^s \cap \mathcal{G}'_p$ .*

**Proof.** First, the space  $\mathcal{M}^s$  is invariant with respect to the following action of  $\mathcal{H}$ :

$$h \circ L = h L s(h)^{-1}.$$

Indeed, let  $L = v s^{-1} u; v, u \in \mathcal{N}$  be an element of  $\mathcal{M}^s$ . Then

$$h \circ L = h v h^{-1} h s^{-1} s(h)^{-1} s(h) u s(h)^{-1} = h v h^{-1} s^{-1} s(h) u s(h)^{-1}. \tag{49}$$

The right-hand side of the last equality belongs to  $\mathcal{M}^s$ , because  $\mathcal{H}$  normalizes  $\mathcal{N}$ .

Using definition (48) of  $\mu_{\mathcal{N}}$  and formula (36) for the factorization problem we can describe the level surface  $\mu_{\mathcal{N}}^{-1}(f)$  as follows:

$$\mu_{\mathcal{N}}^{-1}(f) = \{h_+^{D_p} f n_-^{-1} s(h_+^{D_p})^{-1} = h_+^{D_p} \circ (f n_-^{-1}) | n_- \in \mathcal{N}, h_+ \in \mathcal{H}\}.$$

Hence  $\mu_{\mathcal{N}}^{-1}(f) \subset \mathcal{M}^s$ . The dimension count shows that  $\mu_{\mathcal{N}}^{-1}(f)$  is open in  $\mathcal{M}^s$ .

This concludes the proof. □

By Theorem 14, the image of the level surface  $\mu_{\mathcal{N}}^{-1}(f)$  under the canonical projection  $\mathcal{G}'_p \rightarrow \mathcal{G}'_p/\mathcal{N}$  is a Poisson submanifold in the quotient  $\mathcal{G}'_p/\mathcal{N}$ . For the reasons which will be explained later we denote this manifold by  $\mathcal{G}'_p/(F, \chi_f)$ . Proposition 15 implies that this manifold is open in  $\mathcal{M}^s/\mathcal{N}$ . Actually one can show that  $\mathcal{G}'_p/(F, \chi_f)$  is an open dense subset in  $\mathcal{M}^s/\mathcal{N}$ .

Now we want to describe the reduced manifold  $\mathcal{G}'_p/(F, \chi_f)$  in terms of constraints. To define the constraints we need the notion of matrix coefficients of the group  $\bar{N}$  which is the target space of the map  $\mu_{\mathcal{N}}$ .

Observe that one can define regular functions on  $\bar{N}$ . Indeed, let  $\varphi$  be a regular function on  $\bar{N}$ . It may be viewed as a function  $\varphi' : \bar{N} \rightarrow \mathbb{C}((z))$ . For if  $L = L(z) \in \bar{N}$  then  $\varphi'(L) = \varphi(L(z))$ . The coefficients of the Laurent series  $\varphi(L(z))$  are well defined regular functions on  $\bar{N}$ . In particular, one can define matrix coefficients of  $\bar{N}$ .

It turns out that the constraints for the reduction, i.e. the matrix coefficients of  $\mu_{\mathcal{N}}$ , are of the first class.

**Theorem 16.**

1. *Matrix coefficients of the map  $\mu_{\mathcal{N}}$  form a Poisson subalgebra  $F$  in the Poisson algebra  $C^\infty(\mathcal{G}'_p)$ .*
2. *Define a map  $\chi_f : F \rightarrow \mathbb{C}$  by  $\chi_f(\mu_{\mathcal{N}}) = f$ , where  $\mu_{\mathcal{N}}$  should be viewed as a matrix of regular functions on  $\mathcal{G}'_p$ , the map  $\chi_f$  is applied to the matrix coefficients of  $\mu_{\mathcal{N}}$ . Then  $\chi_f$  is a character of the Poisson algebra  $F$ .*

**Proof.**

1. By part (1) of Theorem 14 the algebra  $\mu_{\mathcal{N}}^*(C^\infty(\bar{\mathcal{N}}))$  is a Poisson subalgebra in  $C^\infty(\mathcal{G}'_p)$ . We have to verify that the pullbacks of the matrix coefficients of  $\bar{\mathcal{N}}$  form a Poisson subalgebra in  $C^\infty(\mathcal{G}'_p)$ .

We shall use gradients of a function  $\varphi \in C^\infty(\mathcal{G}'_p)$  with respect to the  $\mathcal{G}^*$  group structure on the set  $\mathcal{G}'_p$ :

$$\begin{aligned} \langle X, \nabla\varphi(L_+, L_-) \rangle &= \left( \frac{d}{ds} \right)_{s=0} \varphi(e^{sX_+} L_+, e^{sX_-} L_-), \quad \langle X, \nabla'\varphi(L_+, L_-) \rangle \\ &= \left( \frac{d}{ds} \right)_{s=0} \varphi(L_+ e^{sX_+}, L_- e^{sX_-}), \quad X \in \mathfrak{g}. \end{aligned} \tag{50}$$

We define  $Z_\varphi \in \mathfrak{g}$  by the following relation:

$$r_+ Z_\varphi - D_p^{-1} \cdot r_- Z_\varphi = \nabla'\varphi.$$

Then the Poisson bracket on the Poisson submanifold  $\mathcal{G}'_p$  may be represented as follows:

$$\begin{aligned} \{\varphi, \psi\}_{D_p}(L_+, L_-) &= \left\langle AdL_+ \cdot D_p^{-1} \cdot Z_\varphi - AdL_- Z_\varphi, \nabla\psi \right\rangle \\ &\quad - \left\langle \nabla\varphi, AdL_+ \cdot D_p^{-1} \cdot Z_\psi - AdL_- Z_\psi \right\rangle. \end{aligned} \tag{51}$$

This expression leads to the following explicit formula for the Poisson bracket of functions which only depends on  $n_+$  (see (48)):

$$\begin{aligned} \{\varphi, \psi\}_{D_p}(L_+, L_-) &= \left\langle \frac{r_-^0 + D_p r_+^0}{r_-^0 - D_p r_+^0} P_h(Adn_+^{-1} \nabla\varphi), P_h(Adn_+^{-1} \nabla\psi) \right\rangle \\ &\quad + \left\langle P_b(Adn_+^{-1} \nabla\varphi), (Adn_+^{-1} \nabla\psi) \right\rangle \\ &\quad - \left\langle (Adn_+^{-1} \nabla\varphi), P_b(Adn_+^{-1} \nabla\psi) \right\rangle, \quad \langle \nabla\varphi(n_+), X \rangle \\ &= \left( \frac{d}{ds} \right)_{s=0} \varphi(e^{sX} n_+) \quad \text{for every } X \in \bar{n}. \end{aligned} \tag{52}$$

Clearly, such functions form a Poisson subalgebra.

Let  $\varphi$  be a matrix coefficient of the map  $\mu_{\mathcal{N}}$ . Then for every  $X \in \bar{n}$   $\langle \nabla\varphi(n_+), X \rangle$  is an element of  $F$  since  $(d/ds)_{s=0} \varphi(e^{sX} n_+)$  is a linear combination of matrix coefficients. Therefore, for  $\varphi, \psi \in F$  the right-hand side of (52) is an element of  $F$ .

2. We have to show that the Poisson bracket (52) vanishes when restricted to the surface  $\mu_{\mathcal{N}}^{-1}(f)$ .

Recall that for every  $X \in \mathfrak{n}$  the action of the corresponding vector field  $\hat{X}$  is given by formula (46):

$$L_{\hat{X}}\varphi = \langle Ad(\pi_{K^+} \mu)(\theta_{K^*}), X \rangle (\bar{\mu}_*(\xi_\varphi)). \tag{53}$$



Now, let  $\varphi = \varphi(n_+)$ . The surface  $\mu_{\mathcal{N}}^{-1}(f)$  is stable, at least locally, under the gauge action of  $\mathcal{N}$ . This implies  $L_{\hat{X}}\varphi|_{\mu_{\mathcal{N}}^{-1}(f)} = 0$ . Therefore,  $\bar{\mu}_*(\xi_\varphi)|_{\mu_{\mathcal{N}}^{-1}(f)} = 0$ , i.e. the Poisson brackets of every function  $\varphi = \varphi(n_+)$  with the matrix coefficients of  $\mu_{\mathcal{N}}$  vanish on the constraint surface. This completes the proof.  $\square$

**Remark 3.** According to Theorem 16, the pair  $(F, \chi_f)$  is a system of first class constraints for the Poisson manifold  $\mathcal{G}'_p$ . Using Dirac's technique (see [6]), we can define the reduced manifold  $\mathcal{G}'_p/(F, \chi_f)$  with the help of these constraints.

### 3.4. Deformation of the Poisson structure

In this section we show that it is possible to use different bialgebra structures to perform the Drinfeld–Sokolov reduction. First we prove that the Poisson manifolds  $\mathcal{G}_{p,\theta}$  are isomorphic for different  $\theta$ . For each  $\theta$  we describe the reduced space  $\mathcal{G}'_p/(F, \chi_f)$  using the corresponding Poisson manifold  $\mathcal{G}'_{p,\theta}$ . Finally, we consider an important case associated with the bialgebra structure related to the new Drinfeld realization of affine quantum groups.

Let  $\theta, \theta' \in \text{End } h$  be two unitary endomorphisms such that  $\det(\theta - 1) \neq 0, \det(\theta' - 1) \neq 0$ . Suppose that they commute with  $D_p$  and with each other. According to the results of Section 3.1, every such endomorphism defines a factorizable bialgebra structure on  $g$ . As a consequence, we obtain two Poisson manifolds  $\mathcal{G}_{p,\theta}, \mathcal{G}_{p,\theta'}$  equipped with Poisson group  $\mathcal{G}$ -actions (34). The restrictions of these actions to the submanifolds  $\mathcal{G}'_{p,\theta}, \mathcal{G}'_{p,\theta'}$  possess moment maps (see Proposition 8). The results of [1] imply that the Poisson manifolds  $\mathcal{G}'_{p,\theta}, \mathcal{G}'_{p,\theta'}$  are isomorphic. Moreover, the isomorphism is given by the gauge action (34), where  $g$  depends on  $L$ . We can define  $g(L)$  more precisely.

Let  $L$  be an element of  $\mathcal{G}'_{p,\theta}$ . From (3) it follows that the factorization (30) for the manifold  $\mathcal{G}'_{p,\theta}$  amounts to the relations:

$$L = L_+^{D_p} L_-^{-1}, \quad L_\pm = h_\pm n_\pm, \quad h_\pm \in \mathcal{H}, \quad n_+ \in \bar{\mathcal{N}}, \quad n_- \in \mathcal{N}, \quad h_\pm = e^{\theta' r_\pm^0 X}, \quad X \in h. \quad (54)$$

**Proposition 17.** Let  $A \in \text{End } h$  be an endomorphism commuting with  $\theta, \theta'$  and  $D_p$ . The map

$$\mathcal{G}'_{p,\theta} \rightarrow \mathcal{G}'_{p,\theta'} : L \mapsto t^{D_p} L t^{-1} = L', \quad t = e^{AX}, \quad (55)$$

where  $X$  is given by (54), is an isomorphism of the Poisson manifolds if and only if  $A$  satisfies the equation:

$$AA^* \frac{D_p - 1}{\theta' r_-^0 - D_p^\theta r_+^0} + A - A^* = \theta' r_+^0 - \theta r_+^0. \quad (56)$$

Let  $L' = L_+^{D_p} L'_-^{-1}, L'_\pm = h'_\pm n'_\pm, h'_\pm \in \mathcal{H}, n'_+ \in \bar{\mathcal{N}}, n'_- \in \mathcal{N}$  be the factorization (54) of  $L'$  as an element of the manifold  $\mathcal{G}'_{p,\theta'}$ . Then in terms of the components  $h'_\pm, n'_\pm$  the map (55) has the form:

$$h'_\pm = e^{\theta' r_\pm^0 Y}, \quad Y = \frac{D_p A - A + D_p^\theta r_+^0 - \theta r_-^0}{D_p^{\theta'} r_+^0 - \theta' r_-^0} X,$$

$$n'_+ = e^{KX} n_+ e^{-KX}, \quad n'_- = e^{D_p K X} n_- e^{-D_p K X},$$

$$K \in \text{End } \mathfrak{h}, \quad K = \frac{A + \theta r_+^0 - \theta' r_+^0}{D_p^{\theta'} r_+^0 - \theta' r_-^0},$$

where  $n_{\pm}$  and  $X$  are given by (54).

The operator  $K$  satisfies the equation:

$$(1 - D_p) K K^* + D_p K - K^* = \frac{\theta r_+^0 - \theta' r_+^0}{\theta' r_+^0 - D_p^{-1} \theta' r_-^0}.$$

**Proof.** Provided by direct calculation using formula (51) for the Poisson bracket on  $\mathcal{G}'_{p,\theta}$ .  $\square$

The main idea of this proposition is that one can use different Poisson structures for the Drinfeld–Sokolov reduction. From Theorem 16, Remark 3 and the previous proposition for  $\theta' = D_p \cdot s$  we obtain the following

**Proposition 18.** Let  $L \in \mathcal{G}'_{p,\theta}$ ,  $L = L_+^{D_p} L_-^{-1}$  be an element of  $\mathcal{G}'_{p,\theta}$ . Fix an operator  $A$  defined in the previous theorem. Consider the map

$$\mu_{\mathcal{N}}^{\theta,K} : \mathcal{G}'_{p,\theta} \rightarrow \overline{\mathcal{N}}; \quad \mu_{\mathcal{N}}^{\theta,K}(L) = e^{KX} n_+ e^{-KX}$$

which is the composition of the isomorphism (55) and the moment map  $\mu_{\mathcal{N}}$ . Then

1. Matrix coefficients of  $\mu_{\mathcal{N}}^{\theta,K}$  form a Poisson subalgebra  $F^{\theta,K}$  in  $C^\infty(\mathcal{G}'_{p,\theta})$ .
2. The map  $\chi_f^{\theta,K} : F^{\theta,K} \rightarrow \mathbb{C}$  defined by  $\chi_f^{\theta,K}(\mu_{\mathcal{N}}^{\theta,K}) = f$  is a character of the Poisson algebra  $F^{\theta,K}$ . Thus  $(F^{\theta,K}, \chi_f^{\theta,K})$  is a system of the first class constraints.
3. The reduced Poisson manifold  $\mathcal{G}'_{p,\theta}/(F^{\theta,K}, \chi_f^{\theta,K})$  is isomorphic to  $\mathcal{G}'_p/(F, \chi_f)$ .

Now we shall describe the reduced space  $\mathcal{G}'_p/(F, \chi_f)$  using the bialgebra structure related to the “new Drinfeld realization” of affine quantum groups [8]. Recall that this bialgebra structure is factorizable, the corresponding r-matrix is given by

$$D_r = P_{\bar{n}} - P_n + D r^0, \tag{57}$$

where

$$D r^0 = P_+^0 - P_-^0,$$

and  $P_{\pm}^0$  are the projection operators onto  $z\mathfrak{h}[[z]]$  and  $z^{-1}\mathfrak{h}[[z^{-1}]]$ , respectively, in the direct sum

$$h = z^{-1}\mathfrak{h}[[z^{-1}]] + \mathfrak{h} + z\mathfrak{h}[[z]]. \tag{58}$$

Denote by  $\mathcal{G}_{p,D}$  the manifold  $\mathcal{G}$  equipped with Poisson bracket (28), where  $r = D_r$ ,  $\sigma = D_p$ . The previous proposition cannot be directly applied to the manifold  $\mathcal{G}_{p,D}$ , since the r-matrix  $D_r$  is not of form (33). But it may be obtained by a limit procedure from an r-matrix of this type.

Indeed, consider the unitary automorphism  $\theta \in \text{End } \mathfrak{h}$  defined by  $(\theta h)(z) = h(uz)$ ,  $u \in \mathbb{C}, |u| < 1$ . Then the kernel of the ‘‘Cartan’’ component  ${}^\theta r^0$  of the corresponding  $r$ -matrix (33) is

$${}^\theta r^0(z) = \sum_{n=-\infty}^{\infty} t \frac{1 + u^n}{1 - u^n} z^n, \tag{59}$$

where  $t \in \mathfrak{h} \otimes \mathfrak{h}$  is the Casimir element of  $\mathfrak{h}$ , etc. Consider the limit  $u \rightarrow 0$ . The only part of the  $r$ -matrix depending on  $u$  is  ${}^\theta r^0$ . From formula (59) it follows that the limit of  ${}^\theta r^0$  is a well-defined skew-symmetric operator on  $\mathfrak{h}$  which coincides with  ${}^D r^0$ . Therefore, the  $r$ -matrix  ${}^\theta r$  degenerates into  ${}^D r$ .

By the limit procedure from Proposition 18 we get

**Theorem 19.** *Let  $L \in \mathcal{G}'_{p,D}$  be an element of the manifold  $\mathcal{G}'_{p,D}$  factorized as in (30):*

$$L = L_+^{D_p} L_-^{-1}, L_{\pm} = h_{\pm} n_{\pm}, h_{\pm} \in \mathcal{H}, n_+ \in \overline{\mathcal{N}}, n_- \in \mathcal{N}, h_{\pm} = e^{D_r^0 X}, X \in \mathfrak{h}.$$

*Define the map  $\mu_{\mathcal{N}}^{D,K} : \mathcal{G}'_{p,D} \rightarrow \overline{\mathcal{N}}$  by*

$$\mu_{\mathcal{N}}^{D,K}(L) = e^{KX} n_+ e^{-KX}, \tag{60}$$

where  $K \in \text{End } \mathfrak{h}$  is an endomorphism commuting with  $s, D_p$  and satisfying the equation:

$$(1 - D_p)KK^* + D_pK - K^* = -\frac{D_p s}{1 - s} P_+^0 - \frac{1}{1 - s} P_-^0 + \frac{1}{2} \frac{1 + s}{1 - s} P_0, \tag{61}$$

$P_0$  is the projection operator onto  $\mathfrak{h}$  in the direct sum (58).

Then

1. Matrix coefficients of  $\mu_{\mathcal{N}}^{D,K}$  form a Poisson subalgebra  $F^{D,K}$  in  $C^\infty(\mathcal{G}'_{p,D})$ .
2. The map  $\chi_f^{D,K} : F^{D,K} \rightarrow \mathbb{C}$  defined by  $\chi_f^{D,K}(\mu_{\mathcal{N}}^{D,K}) = f$  is a character of the Poisson algebra  $F^{D,K}$ . Thus  $(F^{D,K}, \chi_f^{D,K})$  is a system of the first class constraints.
3. The reduced Poisson manifold  $\mathcal{G}'_{p,D}/(F^{D,K}, \chi_f^{D,K})$  is isomorphic to  $\mathcal{G}'_p/(F, \chi_f)$ .

Note that the Eq. (61) does not contain elliptic functions in the right-hand side.

Similarly to this theorem, we obtain the following description of the reduction for constant loops (see Corollary 10).

**Corollary 20.** *Denote by  $\mathcal{G}'_D$  the manifold  $G$  equipped with Poisson bracket (28), where  $r$  is the restriction of the  $r$ -matrix  ${}^D r$  to constant loops and  $\sigma = id$ . Let  $L \in \mathcal{G}'_D$  be an element of the manifold  $\mathcal{G}'_D$  factorized as in (30):  $L = L_+ L_-^{-1}, L_{\pm} = h_{\pm} n_{\pm}, h_{\pm} \in H, n_+ \in \overline{\mathcal{N}}, n_- \in \mathcal{N}, h_{\pm} = e^{D_r^0 X}, X \in \mathfrak{h}$ .*

*Define the map  $\mu_{\mathcal{N}}^{D,K} : \mathcal{G}'_D \rightarrow \overline{\mathcal{N}}$  by*

$$\mu_{\mathcal{N}}^{D,K}(L) = e^{KX} n_+ e^{-KX}, \tag{62}$$

where  $K \in \text{End } \mathfrak{h}$  is an endomorphism commuting with  $s$  and satisfying the equation:

$$K - K^* = \frac{1}{2} \frac{1 + s}{1 - s}. \tag{63}$$

Then

1. Matrix coefficients of  $\mu_N^{D,K}$  form a Poisson subalgebra  $F^{0,K}$  in  $C^\infty(G'_D)$ .
2. The map  $\chi_f^{0,K} : F^{0,K} \rightarrow \mathbb{C}$  defined by  $\chi_f^{0,K}(\mu_N^{D,K}) = f$  is a character of the Poisson algebra  $F^{0,K}$ . Thus  $(F^{0,K}, \chi_f^{0,K})$  is a system of the first class constraints.
3. The reduced Poisson manifold  $G'_D/(F^{0,K}, \chi_f^{0,K})$  is isomorphic to an open Poisson submanifold in  $S$ .

#### 4. Discussion

In conclusion we briefly discuss quantization of the Poisson–Lie version of the Drinfeld–Sokolov reduction.

First observe that the affine quantum group  $U_q(\hat{\mathfrak{g}})$  with central charge  $q^c = p$  is a quantization of the Poisson manifold  $\mathcal{G}'_{p,D}$  (see [15,18]). The algebra  $U_q(\hat{\mathfrak{g}})$  contains the quantum group  $U_q(\mathfrak{g})$  which is a quantization of the Poisson manifold  $G'_D$  (see [10], Section 2, [22], Section 3).

For simplicity we shall consider the reduction for constant loops (see Corollary 20) in detail. Let us examine map (62). The group  $\bar{N}$  is unipotent and may be identified with its Lie algebra  $\bar{\mathfrak{n}}$  by means of the exponential map. Let  $f_i, i = 1, \dots, l$  be the simple root generators of  $\bar{\mathfrak{n}}$ . An element  $n_+ \in \bar{N}$  may be written as  $e^{\sum_{i=1}^l f_i \varphi_i} + \psi$ , where  $\varphi_i \in \mathbb{C}$  and  $\psi$  is a term of higher order with respect to the principal grading of  $\bar{\mathfrak{n}}$ .

Using the notation of Corollary 20, we can expand  $X \in \mathfrak{h}$  with respect to the basis of root generators as follows:  $X = \sum_{i=1}^l H_i \psi_i$ . Consider  $\varphi_i, \psi_i, i = 1, \dots, l$  as functions  $\varphi_i, \psi_i : \bar{B} \rightarrow \mathbb{C}, \varphi_i(L_+) = \varphi_i, \psi_i(L_+) = \psi_i$ . Then the map (62) induces the following mapping of functions:

$$(\mu_N^{D,K})^*(\varphi_i) = e^{\sum_{j=1}^l \langle \alpha_j, K H_j \rangle \psi_j} \varphi_i. \tag{64}$$

The right-hand side of (64) is an element of the algebra of constraints.

Functions  $\varphi_i, \psi_i$  correspond to simple positive root generators and the fundamental weight generators of  $U_q(\mathfrak{g})$ , respectively. In [25] we have shown that transformation (64) has an exact quantum counterpart (see formula (19) in [25]). Quantum constraints defined in [25] form a subalgebra in  $U_q(\mathfrak{g})$ . This subalgebra possesses a character which is a quantum counterpart of  $\chi_f^{0,K}$ . Eq. (63) appears in the quantum case as well (Eq. (18) in [25]). This allows us to establish an exact correspondence between the classical and the quantum pictures. A detailed analysis of the quantum reduction is contained in [27], Chapter 4.

A similar situation is observed in the affine case. Matrix coefficients of the components  $X, n_\pm$  introduced in Theorem 19 correspond to the loop generators of  $U_q(\hat{\mathfrak{g}})$  in Drinfeld’s “new realization” (see [8,15]). Quantum analogues of map (60), Eq. (61) and the character  $\chi_f^{D,K}$  are defined in terms of Drinfeld’s “new realization” as well (see formula (23), Eq. (33) and discussion after Proposition 8 in [25]).

Finally, we remark that a quantization of the reduced space  $\mathcal{G}'_{p,D}/(F^{D,K}, \chi_f^{D,K})$  may be obtained by applying the homological reduction procedure proposed in [26] for arbitrary systems of the first-class constraints. This program will be realized in a subsequent paper.

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